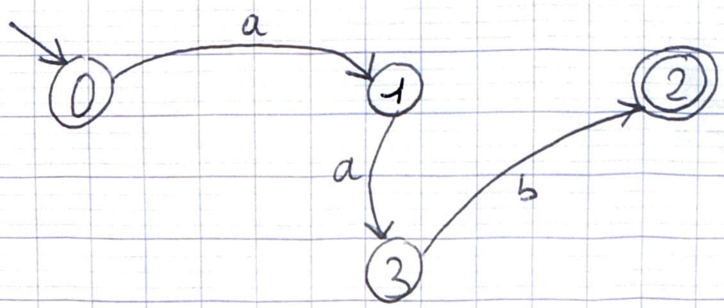
Théorie des langages
Correction du TD n° 2

Exercice 1

Question (a)

$\{w \in \Sigma^* \mid w = aab\}$



Question (b)

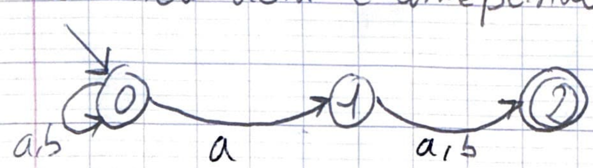
$\{w \in \Sigma^*\}$

Question (c)

$\{w \in \Sigma^* \mid w = uaba,\ u \in \Sigma^*\}$

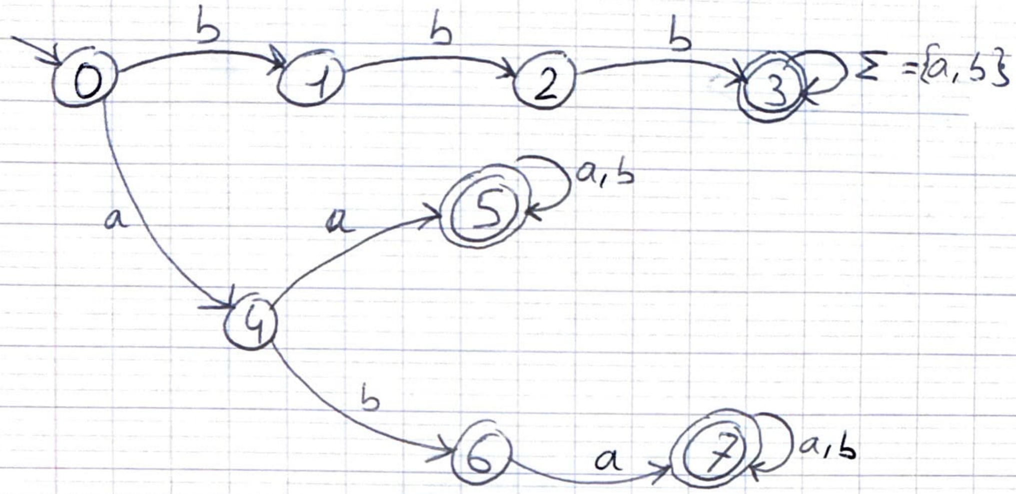
Question (d)

les mots dont l'antépénultième lettre est un a.



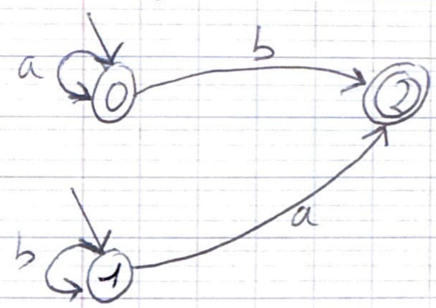
Question (e)

$\{aba, aa, bbb\}^*$



Question (f)

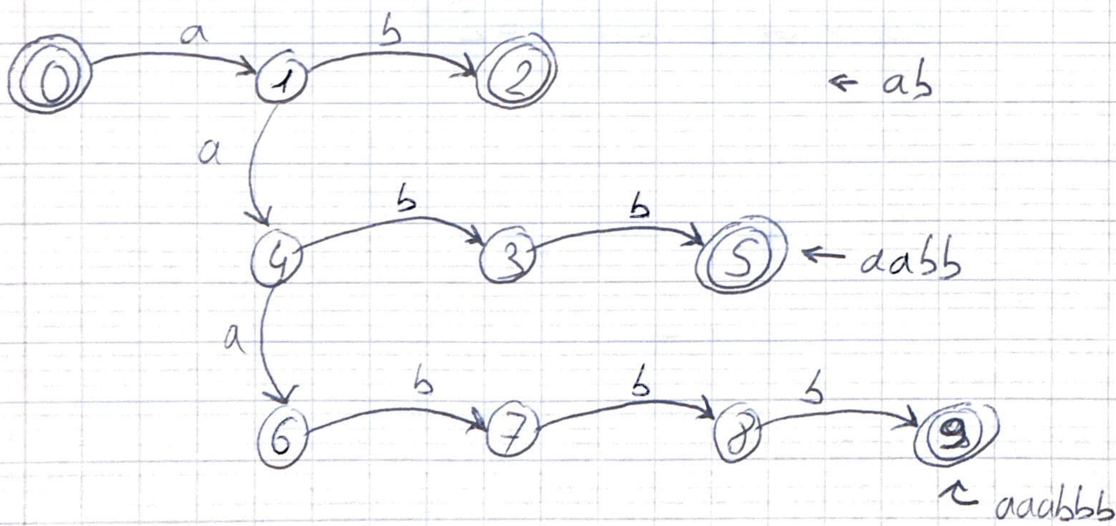
$a^*b + b^*a$



en partant de $Q_0 = a^*b$
en partant de $Q_1 = b^*a$

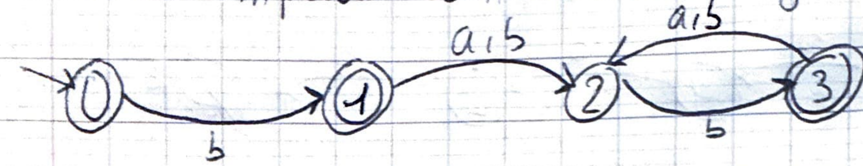
Question (g)

$\{a^n b^n \mid 0 \leq n \leq 3\}$



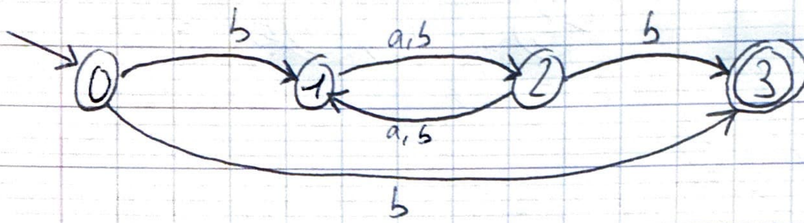
Question (h)

mots impairs commencent et finissent par b.



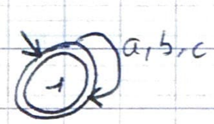
Question (i)

$\{ w \in \Sigma^* \mid \forall a_i \in w,\ a_i = a$
$a_{i-1} = a_{i+1} = b \}$



Exercice 2

On considère l'automate suivant :



$\mathcal{A}_1$

On a $L(\mathcal{A}_1) = L_1$

$L_1 = aL_1 + bL_1 + cL_1 + \varepsilon$

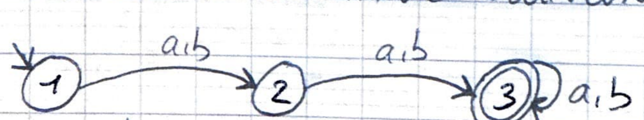
$= \underbrace{(a+b+c)}L_1 + \varepsilon$

$= \Sigma L_1 + \varepsilon$ règle d'Arden

$= \Sigma^* \varepsilon = \Sigma^*$

Ainsi on a $L(\mathcal{A}_1) = \Sigma^*$

On considère l'automate suivant :



$\mathcal{A}_2$

On a alors $L(\mathcal{A}_2) = L_1$

$$\begin{cases} L_1 = (a+b)L_2 = \Sigma L_2 \\ L_2 = (a+b)L_3 = \Sigma L_3 \\ L_3 = (a+b)L_3 + \varepsilon = \Sigma L_3 + \varepsilon \end{cases}$$

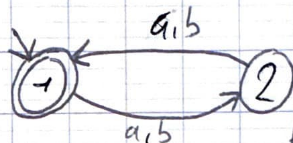
On a $L_3 = \Sigma L_3 + \varepsilon$ règle d'Arden
$= \Sigma^* \varepsilon$
$= \Sigma^*$

$L_2 = \Sigma \Sigma^*$

$L_1 = \Sigma \Sigma \Sigma^* = \Sigma^2 \Sigma^*$

d'où $L(\mathcal{A}_2) = \Sigma^2 \Sigma^*$

On considère l'automate suivant :



$\mathcal{A}_3$

Ainsi $L(\mathcal{A}_3) = L_1$

$$\begin{cases} L_1 = (a+b)L_2 + \varepsilon \\ L_2 = (a+b)L_1 = \Sigma L_1 \end{cases}$$

d'où

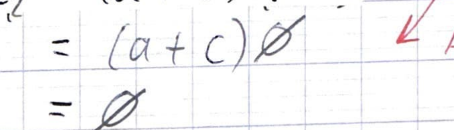
$L_1 = \Sigma \Sigma L_1 + \varepsilon$

$= \Sigma^2 L_1 + \varepsilon$ règle d'Arden

$= (\Sigma^2)^* \varepsilon = (\Sigma^2)^*$

On a alors $L(\mathcal{A}_3) = (\Sigma^2)^*$

On considère l'automate suivant :



$\mathcal{A}_4$

$L(\mathcal{A}_4) = L_1$

$$\begin{cases} L_1 = (a+c)L_1 + bL_2 + \varepsilon \\ L_2 = (a+c)L_2 \end{cases}$$

Alors on a

$L_2 = (a+c)L_2 + \emptyset$ ajout de $\emptyset$ (neutre pour +)

$= (a+c)\emptyset$ Arden

$= \emptyset$

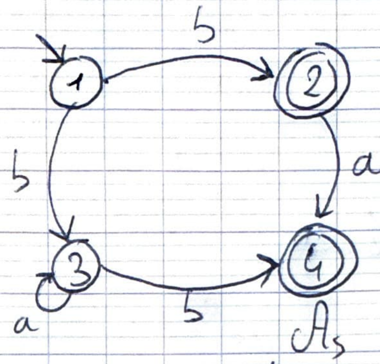
d'où

$L_1 = (a+c)L_1 + \varepsilon \overset{\text{Arden}}{=} (a+c)^* \varepsilon = (a+c)^*$

On se retrouve avec

$L(\mathcal{A}_4) = (a+c)^*$.

On considère l'automate suivant



Ainsi $L(\mathcal{A}_5) = L_1$

$$\begin{cases} L_1 = bL_2 + bL_3 \\ L_2 = aL_4 + \varepsilon \\ L_3 = aL_3 + bL_4 \\ L_4 = \varepsilon \end{cases}$$

On obtient $L_4 = \varepsilon$

d'où

$$\begin{cases} L_1 = bL_2 + bL_3 \\ L_2 = a\varepsilon + \varepsilon = a \\ L_3 = aL_3 + b\varepsilon = aL_3 + b \end{cases}$$

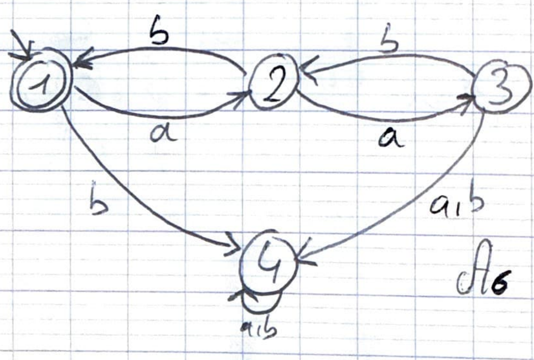
$L_3 = aL_3 + b$
$\quad = a^*b$

$L_2 = a$

$L_1 = ba + ba^*b$

Ainsi on a $L(\mathcal{A}_5) = ba + ba^*b$

On considère l'automate suivant



$L(\mathcal{A}_6) = L_1$

$$\begin{cases} L_1 = aL_2 + bL_4 + \varepsilon \\ L_2 = aL_3 + bL_1 \\ L_3 = bL_2 + (a+b)L_4 \\ L_4 = (a+b)L_4 \end{cases}$$

On peut remplacer $(a+b)$ par Σ.

d'où

$$\begin{cases} L_1 = aL_2 + bL_4 + \varepsilon \\ L_2 = aL_3 + bL_1 \\ L_3 = bL_2 + \Sigma L_4 \\ L_4 = \Sigma L_4 \end{cases}$$

On a $L_4 = \Sigma L_4 + \emptyset$
$= \Sigma^* \emptyset = \emptyset$

d'où

$$\begin{cases} L_1 = aL_2 + b\emptyset + \varepsilon = aL_2 + \varepsilon \\ L_2 = aL_3 + bL_1 \\ L_3 = bL_2 + \Sigma\emptyset = bL_2 \end{cases}$$

Alors

$$L_2 = aL_3 + bL_1 = abL_2 + bL_1$$
$$= (ab)^* bL_1$$
$$L_1 = a(ab)^* bL_1 + \varepsilon$$
$$= (a(ab)^* b)^* \varepsilon = (a(ab)^* b)^*$$

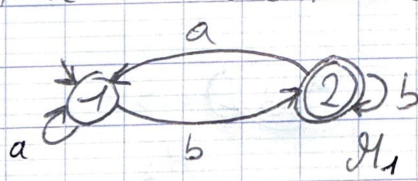
Ainsi $\mathcal{L}(\mathcal{A}_6) = (a(ab)^* b)^*$

Exercice 3

Question (a)

On veut montrer que $\mathcal{L}(\mathcal{M}_1) = \Sigma^* b$

On a l'automate



$\mathcal{L}(\mathcal{M}_1) = L_1$

$$\begin{cases} L_1 = \boxed{aL_1 + bL_2} \\ L_2 = \boxed{aL_1 + bL_2} + \varepsilon \end{cases}$$

Alors on a

$L_2 = L_1 + \varepsilon$

d'où

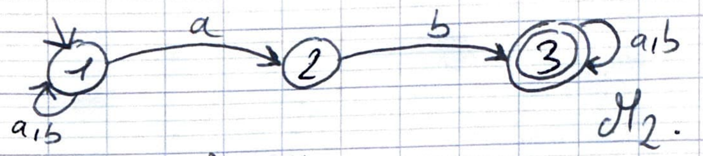
$$L_1 = aL_1 + b(L_1 + \varepsilon)$$
$$= aL_1 + bL_1 + b$$
$$= (a+b)L_1 + b = \Sigma L_1 + b \xrightarrow{\text{Arden}} = \Sigma^* b$$

On retrouve bien $\mathcal{L}(\mathcal{M}_1) = \Sigma^* b$

Question (b)

Mq. $\mathcal{A}_2$ et $\mathcal{A}_3$ sont équivalents

Calcul de $L(\mathcal{A}_2)$.



On a $L(\mathcal{A}_2) = L_1$

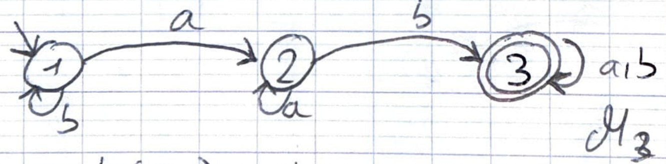
$$\begin{cases} L_1 = (a+b)L_1 + aL_2 = \Sigma L_1 + aL_2 \\ L_2 = bL_3 \\ L_3 = (a+b)L_3 + \varepsilon = \Sigma L_3 + \varepsilon \end{cases}$$

d'où :

$$\begin{cases} L_3 = \Sigma L_3 + \varepsilon \xrightarrow{\text{arden}} = \Sigma^* \varepsilon = \Sigma^* \\ L_2 = bL_3 = b\Sigma^* \\ L_1 = \Sigma L_1 + ab\Sigma^* \xrightarrow{\text{Arden}} = \Sigma^* ab \Sigma^* \end{cases}$$

Ainsi $L(\mathcal{A}_2) = \Sigma^* ab \Sigma^*$

Calcul de $L(\mathcal{A}_3)$



On a $L(\mathcal{A}_3) = L_1$

$$\begin{cases} L_1 = aL_2 + bL_1 \\ L_2 = aL_2 + bL_3 \\ L_3 = (a+b)L_3 + \varepsilon = \Sigma L_3 + \varepsilon \end{cases}$$

On remonte le système

$$\begin{cases} L_3 = \Sigma L_3 + \varepsilon \xrightarrow{\text{Arden}} = \Sigma^* \varepsilon = \Sigma^* \\ L_2 = aL_2 + bL_3 = aL_2 + b\Sigma^* = a^* b \Sigma^* \\ L_1 = bL_1 + a(a^* b \Sigma^*) \\ \quad = b^* a^+ b \Sigma^* \end{cases}$$

On a $L_1 = b^* a^+ b \Sigma^*$
$= b^* a^* a b \Sigma^*$
$= \Sigma^* a b \Sigma^*$

$a^+ = aa^* = a^* a$
$a^* b^* = b^* a^* = \Sigma^*$

d'où $L(\mathcal{M}_3) = \Sigma^* a b \Sigma^*$
et
$$L(\mathcal{M}_2) = L(\mathcal{M}_3)$$